Spherical normal forms of parabolic fixed-points in $\overline{\mathbb{C}}$

Bifurcations and fractal Zeta functions of orbits

Loïc Teyssier (Université de Strasbourg)

May 13^{th}, 2023

Context

Holomorphic dynamical systems on the Riemann sphere $\overline{\mathbb{C}}$

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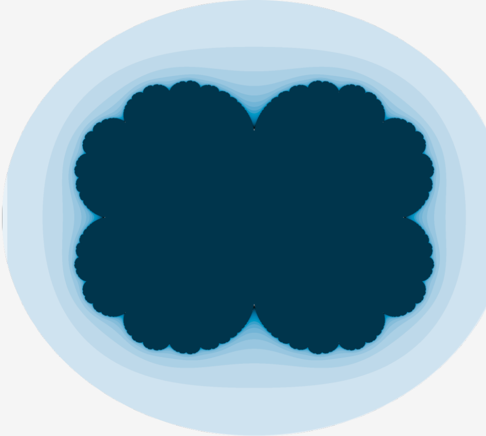

$\mathtt{Julia}\left(z \mapsto z^2 + \frac{1}{4}\right)$

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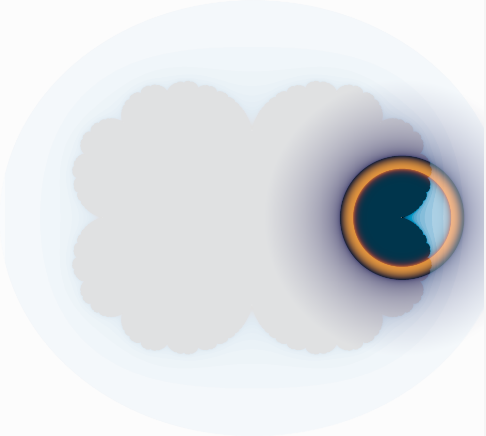

Dynamics of parabolic fixed-points of germs $\Delta \in \mathrm{Diff}\,(\mathbb{C}, 0)$

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   - else: numerous local classes *e.g.*

$$\{z \mapsto z + \cdots\} /_{\mathrm{Diff}(\mathbb{C},0)} \simeq \bigoplus_{\mathbb{N}} \mathrm{Diff}\,(\mathbb{C},0)$$

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- $k \geq 1$ topological invariant: number of attracting petals

$$\mathrm{Parab}_k = \mathrm{Parab} \cap \left(\mathrm{Homeo}^+ (\mathbb{C}, 0)^* \, \mathrm{Parab}_k\right)$$

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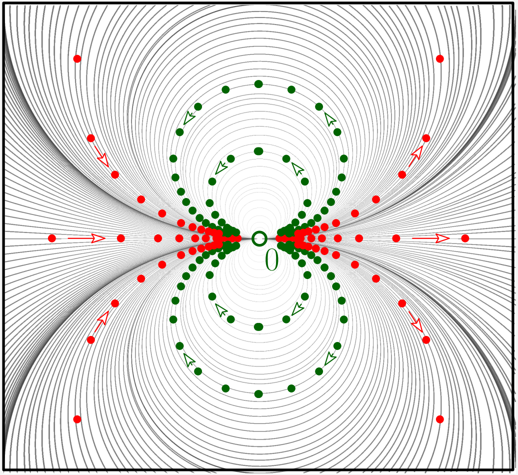

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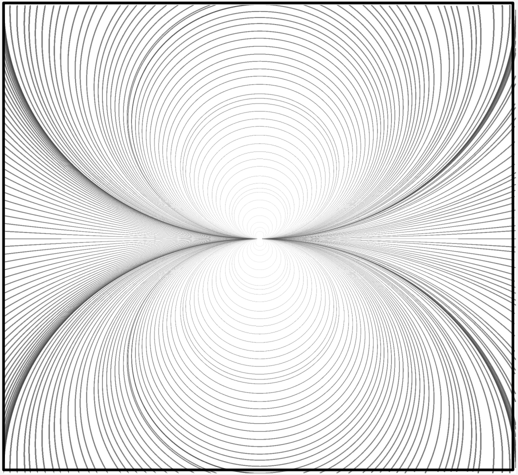
Δ time-1 flow of $z^2 \frac{\partial}{\partial z}$:

$$\dot{x}(t) = x(t)^2 \;\; , \; x(0) = z$$

$$x(1) = \Delta(z)$$



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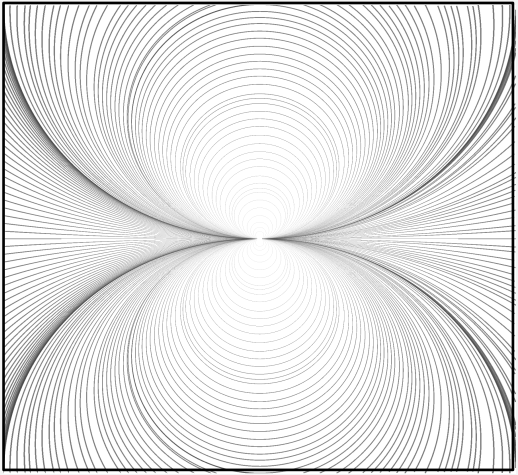

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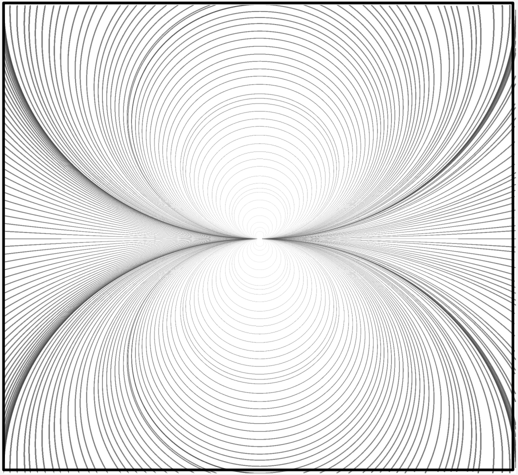

Flow embedding

Definition

$\Delta \in \mathrm{Diff}\,(\mathbb{C}, 0)$ can be **embeded in a flow**
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Answer

No. Example of Baker (1962)

$$\exp -\mathrm{Id}$$

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Lemma

Every $\Delta \in \mathrm{Diff}(\mathbb{C}, 0)$ can be embedded in a a ***formal flow****:*

$$\Delta = \Phi_{\widehat{X}}^1 \quad , \; \widehat{X} \in \mathbb{C}[[z]] \frac{\partial}{\partial z}$$

i.e. the power series

$$\sum_{n \in \mathbb{N}} \frac{t^n}{n!} \left(\widehat{X} \cdot^n \mathrm{Id} \right) =: \Phi_{\widehat{X}}^t$$

converges towards Δ on $(\mathbb{C}, 0)$ for $t := 1$

Flow embedding

Theorem (Écalle, 1975)

For $\Delta = \Phi^1_{\widehat{X}} \in \text{Parab}$ define

$$\Gamma_\Delta := \left\{ t \in \mathbb{C} \ : \ \Phi^t_{\widehat{X}} \text{ converges near } 0 \right\} < (\mathbb{C}, +)$$

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Remark

In general $\Gamma_\Delta = \mathbb{Z}$

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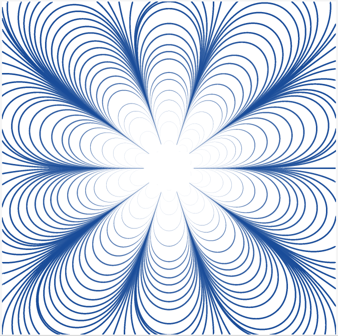
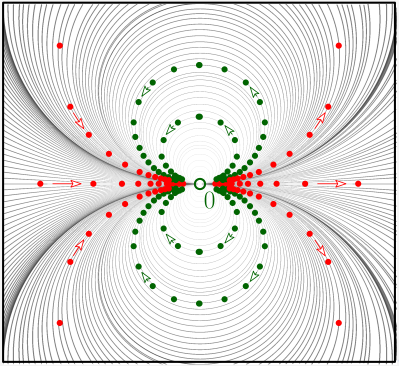


Quotient $\mathcal{P}/_{\mathrm{Diff}(\mathbb{C},0)}$

Heuristics

conformal class of Δ $=$ conformal class of orbits space of Δ

$\longrightarrow$ fundamental domain



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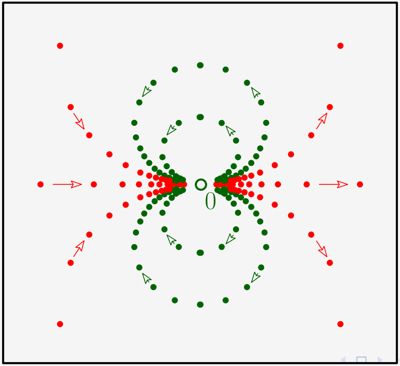

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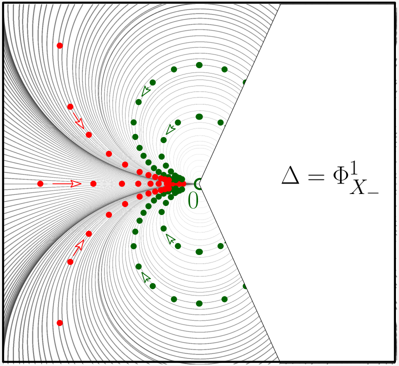

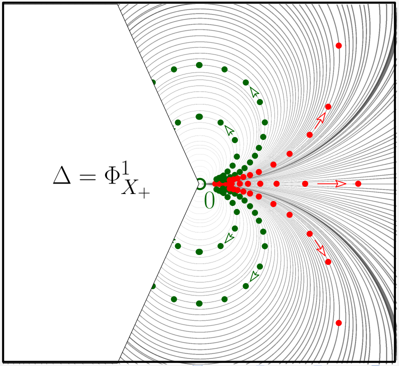

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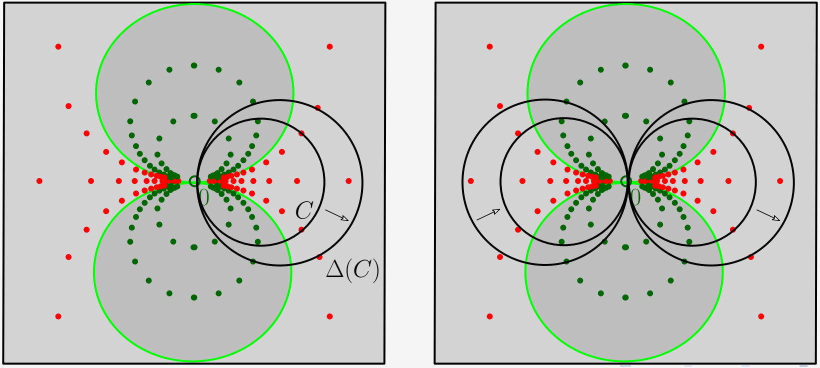

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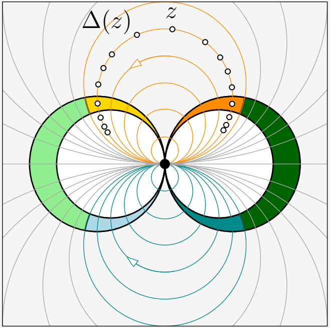

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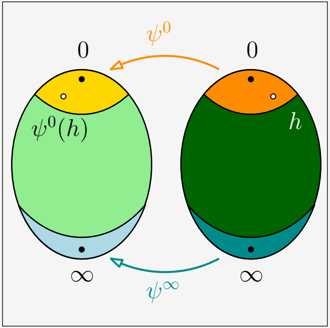

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Theorem (Birkhoff 1939–Écalle 1975–Voronin 1981)

The mapping:

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1. $\psi^{0,\infty}$: horn maps
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- $g \in \text{Centre}(\Delta)$ induces $g^* : \overline{\mathbb{C}} \xrightarrow{\simeq} \overline{\mathbb{C}}$ fixing $\{0, \infty\}$
- $g^* : h \mapsto ch$ for $c \in \mathbb{C}^\times$ hence $g = \Phi_{\widehat{X}}^{\frac{\log c}{2i\pi}}$

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- $\psi\left(h\right) = h\sum_{p\geq0}\alpha_p h^p$ with $\alpha_0 \neq 0$:

$$\alpha_p \neq 0 \Longrightarrow c^p = 1$$

Inverse problem

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Surjectivity of BEV?

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Theorem (Écalle–Voronin)

The mapping

$$\begin{aligned}\mathrm{BEV} \ : \ \mathcal{P}/_{\mathrm{Diff}(\mathbb{C},0)} &\longrightarrow \mathsf{Parab} \times \mathsf{Parab}/_{\mathbb{C}^\times} \\ [\Delta] &\longmapsto \left[\left(\psi^0, \psi^\infty\right)\right]\end{aligned}$$

is bijective

Inverse problem: «abstract» realization

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$$\begin{aligned} H \ : \ V^\pm &\longrightarrow \mathbb{C}^\times \\ x &\longmapsto h = \exp \frac{-2\mathrm{i}\pi}{x} \end{aligned}$$

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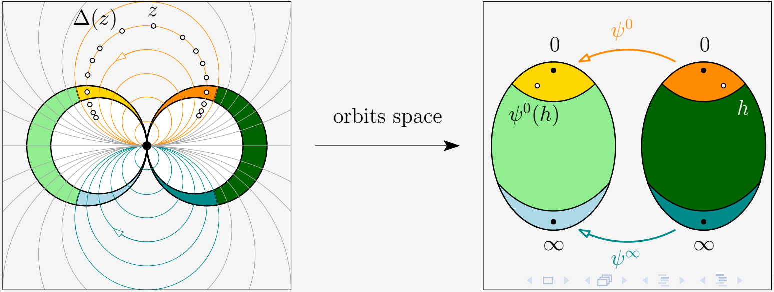
5. Ahlfors-Bers: $V \simeq (\mathbb{C},0)$ together with $\Delta \in \mathcal{P}$

Inverse problem: «abstract» realization

Technical points



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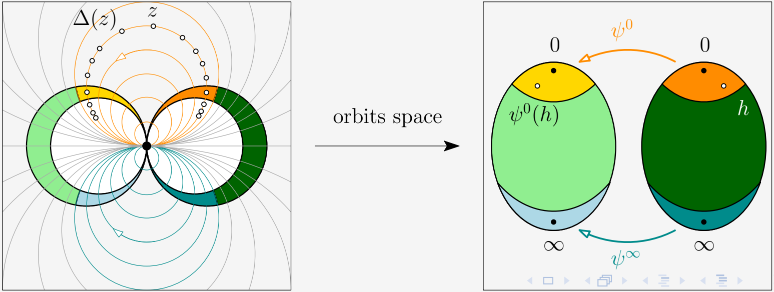
- "The manifold V is obtained by gluing V^+ and V^- in H-space by (ψ^0, ψ^∞)"
  $\longrightarrow$Need to control the size of $H\left(V^+ \cap V^-\right)$
  $\longrightarrow$Constraint on the size of $V^+ \cap V^- \cap (\mathbb{C}, 0)$



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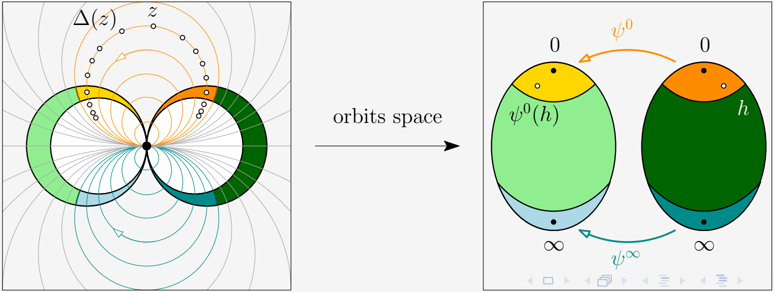
- "Ahlfors-Bers: $V \simeq (\mathbb{C}, 0)$ together with $\Delta \in \mathcal{P}$"
  $\longrightarrow$No control on the «shape» of Δ
  $\longrightarrow$No privileged choice (**normal form**)



Gluing size: rational maps

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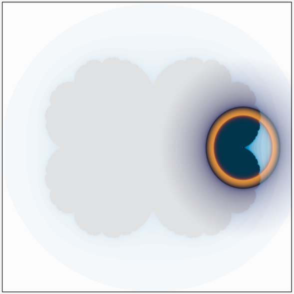
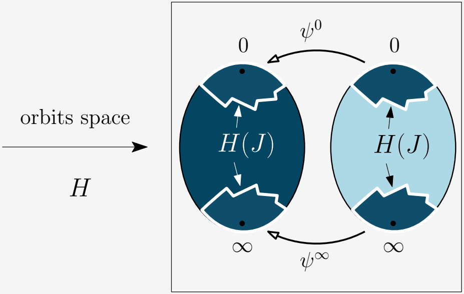
Theorem (Epstein, 1993)

Let R be a rational map, with a parabolic fixed-point at 0 and Julia $(R) \neq \emptyset$. Then $\mathrm{BEV}(R)$ has a continuation frontier.





Spherical normal forms

Spherical normal forms

Theorem

Being given (ψ^0, ψ^∞), for every small enough $\lambda > 0$ there exists a unique power series $F \in z\mathbb{C}[[z]]$ satisfying the next properties. Set

$$X_0(z) := \frac{\lambda z^2}{1+z^2}\frac{\partial}{\partial z} \quad , \; \widehat{X} := \frac{1}{1 + X_0 \cdot F} X_0$$

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1. $\Delta := \Phi^1_{\widehat{X}} \in \mathcal{P}$ *and*

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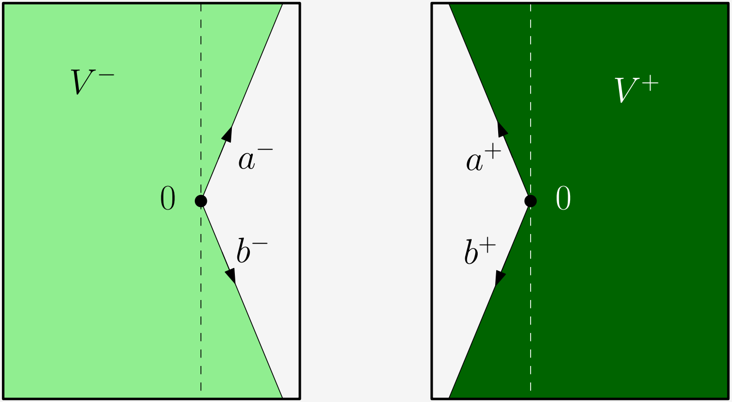
2. *The power series* F *is* 1-*summable with 1-sum* (f^+, f^-) *holomorphic and bounded by* 1 *on the infinite sectors*

$$V^\pm := \left\{ z \neq 0 \;:\; |\arg(\pm z)| < \frac{5\pi}{8} \right\}$$

The infinite sectors



Some innards

- Start with $\left(\psi^0, \psi^\infty\right)$ and write $\psi^{0,\infty}(h) = h \exp \varphi^{0,\infty}(h)$

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- For $f := (f^-, f^+)$ holomorphic on $V^\pm$ and continuous on $\overline{V^\pm}$ define

$$H(z) := \exp\left(-2\mathrm{i}\pi \frac{1-z^2}{\lambda z} + 2\mathrm{i}\pi f^+(z)\right)$$

and $\Lambda(f) = (\Lambda^-, \Lambda^+)$ by

$$\frac{2\mathrm{i}\pi}{\sqrt{z}} \Lambda^\pm(z) := \int_{a^\pm} \frac{\varphi^0(H(\xi))}{\sqrt{\xi}(\xi - z)} \mathrm{d}\xi - \int_{b^\pm} \frac{\varphi^\infty(H(\xi))}{\sqrt{\xi}(\xi - z)} \mathrm{d}\xi$$

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2. $\left\Vert \Lambda\left(f_{2}\right)-\Lambda\left(f_{1}\right)\right\Vert \leq\lambda^{2}C\left(\varphi\right)\left\Vert f_{1}-f_{2}\right\Vert$

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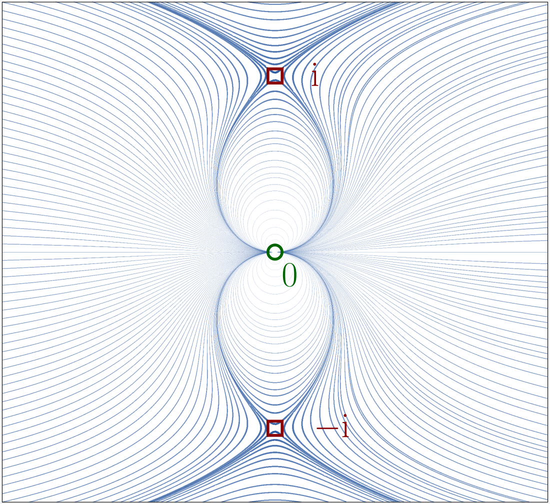
Then:

1. $\Lambda_f^- - \Lambda_f^+ = \begin{cases} \varphi^0 \circ H & \text{on } V^0 \\ \varphi^\infty \circ H & \text{on } V^\infty \end{cases}$
2. $||\Lambda(f_2) - \Lambda(f_1)|| \leq \lambda^2 C(\varphi) ||f_1 - f_2||$

- $\Delta^\pm := \Phi^1_{\frac{1}{1+X_0 \cdot f^\pm} X_0}$ match in $V^+ \cap V^- \Longleftrightarrow f = \Lambda(f)$

A quasi-global dynamics

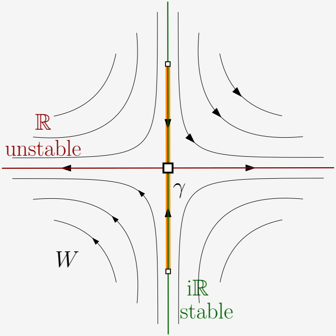
Dynamics of $X_0 = \left(\frac{\lambda \mathrm{Id}}{1-\mathrm{Id}^2}\right)^* \left(x^2 \frac{\partial}{\partial x}\right)$



size of $H\left(V^+ \cap V^-\right) = O\left(e^{-1/\lambda}\right)$

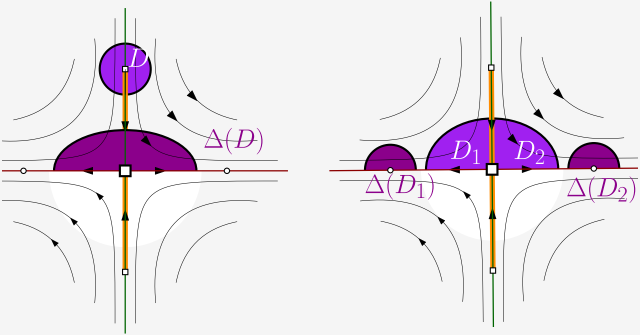
A quasi-global dynamics

Dynamics of a pole $\frac{1}{w}\frac{\partial}{\partial w}$



A quasi-global dynamics

Dynamics of $\Phi^1_{\frac{1}{w}\frac{\partial}{\partial w}}$



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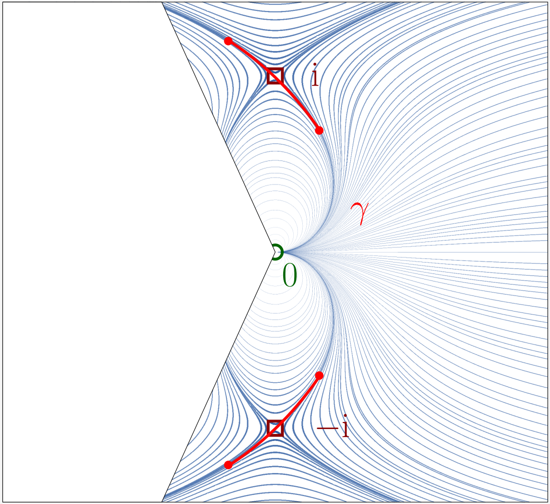

Δ holomorphic and injective on $V^+ \backslash \gamma$

Antipodal dynamics

Δ defines a «companion» dynamics near ∞

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Proposition

$$\mathrm{BEV}_\infty(\Delta) = \mathrm{BEV}_0(\Delta)^{\circ -1}$$

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Proof.

Recall $\Lambda = (\Lambda^-, \Lambda^+)$. Then

$$\Lambda\left(\frac{-1}{z}\right) = -\left(\Lambda^+(z), \Lambda^-(z)\right)$$

□

Antipodal dynamics

Écalle (2005) built **spherical normal forms** with very similar properties

«*As already pointed out, our twisted monomials have much the same behavior at both poles of the Riemann sphere. The exact correspondence has just been described [...] using the so-called antipodal involution: in terms of the objects being produced, this means that canonical object synthesis automatically generates two objects: the "true" object, local at* 0 *and with exactly the prescribed invariants, and a "mirror reflection", local at* ∞ *and with closely related invariants. Depending on the nature of the [...] invariants (verification or non-verification of an "overlapping condition"), these two objects may or may not link up under analytic continuation on the Riemann sphere.*»

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Corollary

Being given $\psi \in$ Parab, there exists a unique normal form $\Delta \in \mathcal{P}$ such that

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Remark

Near ∞ we have

$$\mathrm{BEV}_{\infty}\,(\Delta) = \left(\Delta^{\circ -1}, \psi^{\circ -1}\right)$$